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Non-Riemannian Geometry of Macroscopic Spin Distributions

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The non-Riemannian geometry of macroscopic spin distributions in thermodynamics and ferromagnetism is obtained from the respective partition functions. An expression for the Cartan torsion in terms of the chemical potential is obtained. Analogies with the Einstein–Cartan theory of gravitation are discussed. From the partition function of ferromagnetism a spin–torsion relation analogous to the one obtained in Einstein–Cartan theory is given where piezomagnetic effects are taken into account.

1. INTRODUCTION

Based on early work by Ikeda (1978) on Finsler geometry and thermodynamics, we compute the metric and torsion of a partition function of thermodynamics. A partition function of ferromagnetism is used with Amari's (1962) stress and deformation theory of elastic ferromagnets in terms of Finsler geometry to establish a linear spin–torsion relation (de Sabbata and Sivaram, 1994) which is analogous to the spin–torsion relation obtained in the Einstein–Cartan theory of gravity. An interesting expression between the chemical potential and torsion is obtained by Ikeda's (1977, 1978) method. A relation between torsion and chemical potential had been obtained earlier by Pronin (1985; Kulikov and Pronin, 1993) in the context of gravity with torsion and quantum materials fields. In Pronin's (1985; Kulikov and Pronin, 1993) approach, torsion plays the role of chemical potential in the gravitational interaction of fermions with antifermions.

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2. PARTITION FUNCTION OF THERMODYNAMICS, CHEMICAL POTENTIAL, AND CARTAN'S TORSION

In this section we shall apply the Ikeda metric for thermodynamics (Ikeda, 1978; de Sabbata and Sivaram, 1994)

$$\overline{g(x)} = -\frac{\partial S}{\partial \beta} \tag{1}$$

where the bar over the metric **g** stands for the average, $S \equiv \ln(Z)$ is the entropy, and Z is the partition function. The partition function of a Bose–Einstein gas is

$$Z = e^{\beta\mu} \tag{2}$$

where $\beta \equiv 1/kT$, with k the Boltzmann constant, μ the chemical potential, and T is the absolute temperature. Thus the entropy S in this case is

$$S = \beta\mu \tag{3}$$

Substitution of (3) into (1) yields the metric

$$g(x) = \mu \tag{4}$$

The teleparallel Cartan torsion is

$$S_{ijk} = \partial_{[i}g_{j]k} \tag{5}$$

where S_{ijk} is the torsion tensor.

Substitution of (4) into (5) reads

$$\vec{S} = -\text{grad}\ \mu \tag{6}$$

where $\vec{S}$ is the Cartan torsion vector. Therefore one obtains a nice expression between the torsion and chemical potential. This is not the first time that the chemical potential has been associated with torsion. Starting from a distinct framework Pronin (1985) was able to show that the torsion plays the role of a chemical potential in gravitational interactions between fermions and antifermions.

3. PARTITION FUNCTIONS OF ELASTIC FERROMAGNETS FROM FINSLER GEOMETRY

In this section we shall apply Ikeda's (1977, 1978) method described in the last section to the partition function of elastic ferromagnets in terms of Finsler geometry. The partition function is defined as $(i, j = 1, 2, 3)$

$$Z = e^{\beta^{ij}g_{ij}(x,\xi)} \tag{7}$$

where ξ^i is the magnetic spin and where $g_{ij}(x, \xi)$ is the Finsler metric of a deformed elastic ferromagnet given by

$$g_{ij}(x, \xi) \cong \dot{g}_{ij}(x) - \gamma_{ijk}\xi^k - \gamma_{ijkl}\xi^k\xi^l \tag{8}$$

where γ_{ijk} is the piezomagnetic tensor, γ_{ijkl} is the magnetostriction tensor, and $\dot{g}_{ij}(x)$ is the non-Riemannian metric due to plastic and elastic effects. Substitution of (8) into (7) allows us to obtain an expression for the entropy S,

$$S \equiv \ln(Z) = -[\dot{g}_{ij} - \gamma_{ijk}\xi^k - \gamma_{ijkl}\xi^k\xi^l]\beta^{ij} \tag{9}$$

Thus the thermodynamic torsion is given by

$$S_{ijk} = \partial_{[i}\overline{g_{j]k}} = -\partial_{[i}\frac{\partial S}{\partial \beta^{j]k}} \tag{10}$$

Substitution of (9) into (10) yields

$$\overline{S}_{ijk} = \overline{\dot{S}}_{ijk} - \partial_{[i}(\overline{\gamma}_{j]kl}\overline{\xi^l}) - \partial_{[i}(\overline{\gamma}_{jkls}\overline{\xi^l\xi^s}) \tag{11}$$

To simplify matters, let us consider only piezomagnetic effects ($\gamma_{ijk} \neq 0$, $\gamma_{ijkl} \equiv 0$). Thus the macroscopic spin distribution becomes

$$\overline{S}_{ijk} = \overline{\dot{S}}_{ijk} - [\partial_{[i}\overline{\gamma}_{j]kl}]\overline{\xi^l} \tag{12}$$

Since $\overline{\partial\xi} = \partial\overline{\xi} = 0$ in the elastic ferromagnetic, ξ = const locally, and therefore expression (12) provides a linear spin–torsion relation (Ikeda, 1977) exactly as in the Einstein–Cartan theory of gravity.

Therefore one concludes that statistical mechanics (partition function) applied to thermodynamics and piezomagnetism leads to well-known results in the Einstein–Cartan theory of gravitation. This is in agreement with Kondo and Amari's (1961) idea that unification among electromagnetism, plasticity, and general relativity can be obtained via statistical mechanics. In particular, our results agree with Kondo's (1962) idea that non-Riemannian geometry can be obtained from Finsler geometry by statistical methods.

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